SOLUTIONS OF PROBLEMS IN NUMBER THREE.

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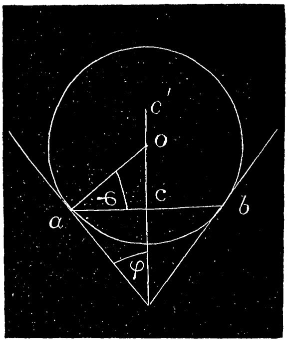
167. "A solid sphere rolls down a trough formed by two planes which make with each other an angle 2φ. Find an expression for the time when the inclination of the trough to the horizon is θ."

SOLUTION BY PROF. C. M. WOODWARD, WASHINGTON UNIV., ST. LOUIS, MO.

The instantaneous axis of the body is the line ab joining the two points of contact. $OC = R\sin\varphi$, and since $OC.OC' = \frac{2}{5}R^2 =$ the square of the radius of gyration, C' is distant from the centre $OC' = \dfrac{2R}{5\sin\varphi}$. Now the whole mass of the sphere may be supposed to be at C and C'. C has no motion, the mass at C' moves freely under the influence of gravity as if sliding down a smooth inclined plane. Its acceleration is therefore $g\sin\theta$. The linear acceleration of the centre will therefore be



$$\frac{R\sin\varphi\,.\,g\sin\theta}{R\sin\varphi+(2R\div5\sin\varphi)}=\frac{5\sin^2\varphi}{5\sin^2\varphi+2}.g\sin\theta.$$

Since $s=\frac{1}{2}.\dfrac{5g\sin^2\varphi\sin\theta}{5\sin^2\varphi+2}.t^2$, we have $t=\dfrac{1}{\sin\varphi}\sqrt{\left[\dfrac{(10\sin^2\varphi+4)s}{5g\sin\theta}\right]}$.

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168. "Find the general value of

$$u=\iint\frac{dxdy}{\sqrt{(1+x^2+y^2)^3}}$$

and show that when the limits of x and y are 0 and 1, $u=0.5+$"

SOLUTION BY PROF. H. T. EDDY, CINCINNATI UNIV., CIN., OHIO.

Transforming to polar coordinates,

$$u=\int_0^\zeta\int_0^{x\sec\theta}\frac{rdrd\theta}{\sqrt{(1+r^2)^3}}+\int_\zeta^{\frac{1}{2}\pi}\int_0^{y\operatorname{cosec}\theta}\frac{rdrd\theta}{\sqrt{(1+r^2)^3}}.$$

$$\therefore\ u=\int_0^\zeta d\theta-\int_0^\zeta\frac{\cos\theta d\theta}{\sqrt{(1+x^2-\sin^2\theta)}}+\int_\zeta^{\frac{1}{2}\pi}d\theta-\int_\zeta^{\frac{1}{2}\pi}\frac{\sin\theta d\theta}{\sqrt{(1+y^2-\cos^2\theta)}},$$

where $\zeta=\tan^{-1}(y\div x)$;

$$\therefore\ u=\frac{\pi}{2}-\sin^{-1}\frac{y}{\sqrt{[(x^2+y^2)(1+x^2)]}}-\sin^{-1}\frac{x}{\sqrt{[(x^2+y^2)(1+y^2)]}},$$

$$\therefore \qquad u = \frac{\pi}{2} - \tan^{-1}\frac{\sqrt{(1+x^2+y^2)}}{xy} = \tan^{-1}\frac{xy}{\sqrt{(1+x^2+y^2)}}.$$

To obtain the general integral, add to the result $\varphi(x)+\psi(y)$.

Between the limits 0 and 1 for both variables, $\tan u = 1 \div \sqrt{3}$; therefore $u = \frac{1}{6}\pi = 0.5+$

[Mr. Adcock gets for the general value

$$u = \tfrac{1}{2}\tan^{-1}\left[\frac{[(1+x^2+y^2)^{\frac{1}{2}}+y]^2-x^2+1}{2x}\right] - \tfrac{1}{2}\tan^{-1}\frac{1}{x}$$
$$+ \tfrac{1}{2}\tan^{-1}\left[\frac{[(1+x^2+y^2)^{\frac{1}{2}}+x]^2-y^2+1}{2y}\right] - \tfrac{1}{2}\tan^{-1}\frac{1}{y}.$$

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169. "Base balls are covered by sewing together two dumb-bell shaped pieces of leather. Determine the shape of the pieces so as to reduce the distortion in fitting them to the spherical surface to a minimum."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Conceive at first the covering to consist of three parts, viz., an equatorial belt and two caps at the poles. The belt is formed from a cylinder touching the ball at the equator, and, considering only one hemisphere, let $p =$ height of this cylinder above plane of equator and $r =$ radius of ball (which includes one half the thickness of the leather), then

$$2rp\pi = \text{surface of one half of equatorial belt.} \qquad (1)$$

The caps at the poles are formed from a tangent plane at the poles; they are circles the radius of which let $= q$, then

$$q^2\pi = \text{surface of one cap at the pole.} \qquad (2)$$

There are two extreme cases within which the correct solution of this problem must be comprised, viz:

1. The material is supposed to be absolutely inextensible.
2. The material is supposed extensible to allow being stretched over a curved surface of equal area to its plane area.

In the first case we must assume that the belt and polar cap, being bent over the ball, will meet, or

$$p + q = \tfrac{1}{2}\pi r. \qquad (3)$$

The distortion in this case is the excess of the covering areas over the area to be covered. Denoting this by T we have

$$T = (2rp + q^2 - 2r^2)\pi = \text{minimum}, \qquad (4)$$

or, introducing the condition (3), we have to make

$$2rp + q^2 - 2r^2 + (p+q-\tfrac{1}{2}\pi r)K = \text{min.},$$

K being an undetermined constant.

We have then the two conditions for the minimum (denoting the minimum values of p and q by p_1 and q_1),

$$2r+K=0, \qquad 2q_1+K=0; \qquad \therefore\ q_1=r. \tag{5}$$

$$\therefore\ p_1=\tfrac{1}{2}\pi r-r=0.57080r. \tag{6}$$

In the second case we have the condition

$$2rp\pi+q^2\pi=2r^2\pi \text{ or } 2rp+q^2=2r^2. \tag{7}$$

The distortion in this case is measured by the small belt left uncovered if the equatorial belt and polar caps are bent over the surface of the ball without being stretched. Call it t; then if the arc s is reckoned from the eq'r,

$$t=2r\pi\int_p^{\frac{1}{2}r\pi-q}\cos\frac{s}{r}\,ds=2r^2\pi\left(\cos\frac{q}{r}-\sin\frac{p}{r}\right), \tag{8}$$

or, introducing the condition (7),

$$\cos\frac{q}{r}-\sin\frac{p}{r}+\frac{rp+\frac{1}{2}q^2-r^2}{k^2}=\text{minimum},$$

where k is an undetermined constant.

We have then the conditions for minimum (denoting the resulting values of p and q by p_2 and q_2)

$$-\frac{1}{r}\sin\frac{q_2}{r}+\frac{q_2}{k^2}=0,\ -\frac{1}{r}\cos\frac{p_2}{r}+\frac{r}{k^2}=0;\ \therefore\ \sin\frac{q_2}{r}=\frac{q_2}{r}\cos\frac{p_2}{r}. \tag{9}$$

From (7) we have

$$\frac{p_2}{r}=1-\frac{q_2^{\,2}}{2r^2}\ldots(10,\ \therefore\ \sin\frac{q_2}{r}=\frac{q_2}{r}\cos\left(1-\frac{q_2^{\,2}}{2r^2}\right). \tag{11}$$

From these equations we find

$$q_2=0.95388r,$$
$$p_2=0.54505r.$$

The dumb-bell shaped pieces may be constructed as follows: Draw a parallelogram with the sides $r\pi$ and $2q$ the altitude of which is $2p$. On the sides $2q$ describe semicircles with q as radius; the resulting figure, together with another piece of the same shape, is the best adapted for covering a ball. In the second and all intermediate cases between the first and second case, the circular edge which is to be sewed to the straight edge is slightly shorter; the stitches have therefore to be made closer on the circular edge and in order to secure the most perfect fit it might be necessary to mark the corresponding points beforehand.

Query (proposed at p. 166). "How is the *Rule*, given at page 44, Gillespie's Land-surveying, (5th edition, New York, 1857,) demonstrated?"

[The rule referred to in the above query, as quoted by Dr. Oliver, is as follows:—"When the four sides and the sum of any two opposite angles are